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# Connection between the Loop Variable Formalism and the Old Covariant Formalism for the Open Bosonic String.

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## Abstract

The gauge invariant loop variable formalism and old covariant formalism for bosonic open string theory are compared in this paper. It is expected that for the free theory, after gauge fixing, the loop variable fields can be mapped to those of the old covariant formalism in bosonic string theory, level by level. This is verified explicitly for the first two massive levels. It is shown that (in the critical dimension) the fields, constraints and gauge transformations can all be mapped from one to the other. Assuming this continues at all levels one can give general arguments that the tree S-matrix (integrated correlation functions for on-shell physical fields) is the same in both formalisms and therefore they describe the same physical theory (at tree level).

# 1 Introduction

A manifestly background independent formalism would be a big step towards obtaining a deeper understanding of string theory. In particular issues such as the space time symmetry principle underlying string theory and a fundamental role of strings in the structure of space time might be elucidated if such a formalism were available. Conventionally, most of our understanding of string theory is based on the world sheet theory. One can derive, mathematically, some symmetry transformations of the space-time fields of string theory starting from world sheet properties such as reparametrization invariance or BRST invariance. On the other hand we know that the low energy effective action is the Einstein action for gravity (or possibly it's supersymmetric generalizations). One expects therefore that the sought after symmetry principle would be some generalization of general coordinate invariance. We do not have an understanding of this today.

The loop variable (LV) formalism [1, 2] incorporates gauge invariance without relying on world sheet properties. This is promising from the point of view of making background independence manifest. In fact it was shown recently that within this approach one can put open strings in a curved space-time background. Thus one can obtain gauge invariant and general covariant equations of motion for massive higher spin fields in arbitrarily curved space-times. Another approach to background independence is described in [5, 6].

Another advantage of this formalism is that the gauge transformations have a simple form of space time scale transformations. This is a step in the direction of understanding the space-time symmetry principle underlying string theory. Some speculations on this are contained in [1]

There are also some intriguing connections with M-theory: the loop variable formalism is more conveniently written in one higher dimension, and also the interacting theory seems best described by a free 'band' .

What has not been done so far is to obtain a precise map from this formalism to other formalisms of string theory. The field content of this theory is the same as that of BRST string field theory [7, 8, 9]. Thus one expects that there should be a map at each level between the fields (and their gauge transformations) of the two formalisms. Alternatively, after gauge fixing one can compare LV formalism with the old covariant formalism. This is what is done in this paper. In the critical dimension the physical state constraints guarantee the absence of unphysical states, and therefore unitarity, in the old covariant (OC) formalism. One expects

that at every level one can gauge fix the loop variable fields and map them to the old covariant formalism fields. This is because at the free level the field content uniquely determines the equations and gauge transformations (modulo field redefinitions). In this paper the existence of such a map is verified by explicit construction for the first two massive levels. This map is between the fields and gauge transformations of the two formalisms. The gauge transformations in the loop variable formalism take the simple form  $k(s) \rightarrow k(s)\lambda(s)$ . In the OC formalism it is generated by  $L_{-n}$ . We show that while the LV formalism is gauge invariant in any dimension, only in the critical dimension do the gauge transformations agree with those of the OC formalism. Similarly the constraints in the OC and LV formalism can also be mapped to each other. Assuming this agreement works at higher levels as well one, can argue that by gauge fixing the LV fields, and using the physical state constraints, one obtains the same vertex operator as in the OC formalism and therefore the integrated correlation functions match. Thus the S-matrices should agree and thus they describe the same theory.

The critical dimension,  $D=26$ , enters in a crucial way for the agreement. Away from  $D=26$ , the LV formalism is still gauge invariant. It is not clear whether it can be related to some non critical string theory.

This paper is organized as follows. In Section 2 we describe the level 2 and 3 vertex operators in the OC formalism and also list the constraints and gauge transformations. This is a review of well known results (see for e.g. [10, 11]). In Section 3 we repeat this for the LV formalism. In Section 4 we discuss the gauge fixing and also give the map between the fields and constraints in the two formalisms. In Section 5 we discuss the interacting theory. Section 6 contains some conclusions.

## 2 Old Covariant Formalism

In this section we will discuss the physical states and gauge transformations in the OC formalism for level 2 and level 3. The physical state constraints are given by the action of  $L_{+n}, n \geq 0$  and gauge transformations by the action of  $L_{-n}, n > 0$ . In [3] a closed form expression is given for the following:

$$e^{\sum_n \lambda_{-n} L_{+n}} e^{i \sum_{n \geq 0} k_n \tilde{Y}_n(z)} |0\rangle \quad (2.0.1)$$

where  $\tilde{Y}_n = \frac{\partial_x^n X(z)}{(n-1)!}$ ,  $\tilde{Y}_0 = \tilde{Y} = X$ . We will need it mainly to linear order in  $\lambda_n$  which can be obtained from:

$$e^{-\frac{1}{2} \mathcal{Y}^\tau \lambda \mathcal{Y}} e^{i \sum_{n \geq 0} k_n \tilde{Y}_n(z)} \quad (2.0.2)$$

where  $\mathcal{Y}^T = (.., \tilde{Y}_3, \tilde{Y}_2, \tilde{Y}_1, -ik_0, -ik_1, -2ik_2, -3ik_3, ...)$  and  $\lambda$  is a matrix whose elements are given by:

$$(\lambda)_{m,n} = \lambda_{m+n}$$

(2.0.2) will be used below.

## 2.1 Level 2

### 2.1.1 Vertex operators

The level two vertex operators can be obtained from

$$e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2}|0\rangle = e^{ik_0.X}(\dots - \frac{1}{2}k_1^\mu k_1^\nu \partial X^\mu \partial X^\nu + ik_2^\mu \partial^2 X^\mu + \dots)|0\rangle \quad (2.1.3)$$

### 2.1.2 Action of $L_{\pm n}$

Using (2.0.2) we get

$$\begin{aligned} \exp \left[ \lambda_0 \left( \frac{k_0^2}{2} + ik_1 \tilde{Y}_1 + 2ik_2 \tilde{Y}_2 \right) + \lambda_{-1} (k_1.k_0 + 2ik_2.\tilde{Y}_1) \right. \\ \left. + \lambda_{-2} (2k_2.k_0 + \frac{1}{2}k_1.k_1) + \lambda_1 (ik_1.\tilde{Y}_2 + ik_0.\tilde{Y}_1) \right. \\ \left. + \lambda_2 \left( -\frac{1}{2}\tilde{Y}_1.\tilde{Y}_1 + ik_0.\tilde{Y}_2 \right) \right] e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2}|0\rangle \end{aligned} \quad (2.1.4)$$

We need to extract terms that have two  $k_1$ 's or one  $k_2$  for  $L_{+n}$ . To get gauge transformations  $L_{-1}, L_{-2}$  we need to extract the level two terms: We can read off the various terms:

1.  $\lambda_0 \mathbf{L}_0$  :

$$\lambda_0 \left[ \frac{k_0^2}{2} \left( -\frac{1}{2}k_1^\mu k_1^\nu \partial X^\mu \partial X^\nu + ik_2^\mu \partial^2 X^\mu \right) - k_1^\mu k_1^\nu \partial X^\mu \partial X^\nu + 2ik_2^\mu \partial^2 X^\mu \right] e^{ik_0.X}|0\rangle \quad (2.1.5)$$

2.  $\lambda_{-1} \mathbf{L}_1$  :

$$\lambda_{-1} [k_1.k_0 \quad ik_1^\mu \partial X^\mu + 2ik_2^\mu \partial X^\mu] e^{ik_0.X}|0\rangle \quad (2.1.6)$$

3.  $\lambda_{-2}\mathbf{L}_2$  :

$$\lambda_2[2k_2.k_0 + \frac{1}{2}k_1.k_1] e^{ik_0.X}|0\rangle \quad (2.1.7)$$

4.  $\lambda_1\mathbf{L}_{-1}$  :

$$\lambda_1[ik_1^\mu\partial^2 X^\mu - k_1^\mu k_0^\nu\partial X^\mu\partial X^\nu] e^{ik_0.X}|0\rangle \quad (2.1.8)$$

(It is easy to see that the above is just  $\lambda_1 L_{-1} ik_1^\mu\partial X^\mu e^{ik_0.X}|0\rangle$ )

5.  $\lambda_2\mathbf{L}_{-2}$  :

$$\lambda_2[-\frac{1}{2}\partial X.\partial X + ik_0^\mu\partial^2 X^\mu] e^{ik_0.X}|0\rangle \quad (2.1.9)$$

(This is just  $\lambda_2 L_{-2} e^{ik_0.X}|0\rangle$ )

The  $L_0 = 1$  equation gives the mass shell condition and  $L_1, L_2 V|0\rangle = 0$  and give additional physical state constraints. It is also important to observe that since  $L_n|0\rangle = 0, n \geq -1$ , the constraints given above are equivalent to  $[L_n, V] = 0, n \geq -1$ . For the gauge transformations  $L_{-2}|0\rangle \neq 0$  and it makes a difference whether one defines gauge transformation on the fields as including the action on  $|0\rangle$  as is being done here, or as only acting on the vertex operators :  $[L_{-n}, V]$ . In LV formalism one does not include the action on the vacuum. This has to be accounted for by field redefinitions.

### 2.1.3 Liouville Mode

One can obtain the physical state constraints also by looking at the Liouville mode dependence. The Liouville mode,,  $\rho$  , is related to  $\lambda_n$  at linear order by

$$\frac{d\lambda}{dt} = \rho \quad (2.1.10)$$

where  $\lambda(t) = \sum_n \lambda_n t^{1-n}$  and  $\rho(t) = \rho(0) + t\partial\rho(0) + \frac{t^2}{2}\partial^2\rho(0) + \dots$ . Thus we get

$$\lambda_0 = \rho, \quad \lambda_{-1} = \frac{1}{2}\partial\rho, \quad \lambda_{-2} = \frac{1}{3!}\partial^2\rho, \quad \lambda_{-3} = \frac{1}{4!}\partial^3\rho \quad (2.1.11)$$

This way of looking at the constraints is useful for purposes of comparison with the LV formalism.

Thus

$$e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2} =: e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2} :$$

$$\begin{aligned}
& e^{\frac{1}{2}[k_0^2\langle XX\rangle+2k_1.k_0\langle X\partial X\rangle+k_1.k_1\langle\partial X\partial X\rangle+2k_2.k_0\langle X\partial^2 X\rangle]} \\
& =: e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2} : e^{\frac{1}{2}[k_0^2\rho+2k_1.k_0\frac{1}{2}\partial\rho+k_1.k_1\frac{1}{6}\partial^2\rho+2k_2.k_0\frac{1}{3}\partial^2\rho]} \quad (2.1.12)
\end{aligned}$$

The Liouville mode dependence is obtained using (2.1.11), (2.1.4). This implies  $\langle XX\rangle = \rho$ ,  $\langle X\partial X\rangle = \frac{1}{2}\rho$ ,  $\langle X\partial^2 X\rangle = \frac{1}{3}\partial^2\rho$ ,  $\langle\partial X\partial X\rangle = \frac{1}{6}\partial^2\rho$ . These can be derived by other methods also [16].

In addition to the anomalous dependences, the Liouville mode also enters at the classical level. The vertex operators on the boundary involve covariant derivatives  $\nabla_x$  where  $x$  is the coordinate along the boundary of the world sheet. The vertex operators on the boundary should be  $\int dx V$  where  $V$  is a one dimensional vector vertex operator or  $\int dx \sqrt{g} S$  where  $S$  is one dimensional scalar. Note that  $g_{xx} = g$  (in one dimension) and  $g^{xx} = \frac{1}{g}$ . The simplest vertex operator is thus  $\nabla_x X = \partial_x X$  (since  $X$  is a scalar). Further  $\nabla^x X = g^{xx} \nabla_x X$  and using  $\nabla_x T^x = \frac{1}{\sqrt{g}} \partial_x (\sqrt{g} T^x)$  we get  $\nabla_x \nabla^x X = \frac{1}{\sqrt{g}} \partial_x \sqrt{g} g^{xx} \partial_x X = \frac{1}{\sqrt{g}} \partial_x \frac{1}{\sqrt{g}} \partial_x X$  is a scalar. Thus  $\sqrt{g} \nabla_x \nabla^x X = \partial_x \frac{1}{\sqrt{g}} \partial_x X$  is the vertex operator with two derivatives. One can similarly show that  $\partial_x \frac{1}{\sqrt{g}} \partial_x \frac{1}{\sqrt{g}} \partial_x X$  is the vertex operator with three derivatives. This pattern continues.

The metric on the boundary is induced by the metric on the bulk:

$$g_{xx} = 2 \frac{\partial z}{\partial x} \frac{\partial \bar{z}}{\partial x} g_{z\bar{z}} = g_{z\bar{z}} \quad (2.1.13)$$

Thus if in the conformal coordinates  $g_{z\bar{z}} = e^{-2\rho}$ , we have  $\frac{1}{\sqrt{g}} = e^\rho$ . Thus

$$\int dx \partial X, \quad \int dx e^\rho (\partial^2 X + \partial \rho \partial X), \quad \int dx e^{2\rho} (\partial^3 X + 3\partial^2 X \partial \rho + \partial X \partial^2 \rho), \dots \quad (2.1.14)$$

Or if we remove  $\int dx \sqrt{g}$  we get

$$e^\rho \partial X, \quad e^{2\rho} (\partial^2 X + \partial \rho \partial X), \quad e^{3\rho} (\partial^3 X + 3\partial^2 X \partial \rho + \partial X \partial^2 \rho), \dots \quad (2.1.15)$$

for the vertex operators. The power of  $e^\rho$  now counts the dimension of the unintegrated vertex operators. Inserting this into (2.2.28) we get:

$$\begin{aligned}
& =: e^{ik_0.X+ik_1.e^\rho \partial X+ik_2.e^{2\rho}(\partial^2 X+\partial \rho \partial X)} : e^{\frac{1}{2}[k_0^2\rho+2k_1.k_0\frac{1}{2}\partial\rho+k_1.k_1\frac{1}{6}\partial^2\rho+2k_2.k_0\frac{1}{3}\partial^2\rho]} \\
& \quad (2.1.16)
\end{aligned}$$

This expression gives the complete  $\rho$  dependence to linear order. One can check that the coefficient of  $\lambda_{-1} = \frac{1}{2}\partial\rho$  is  $2ik_2^\mu \partial X^\mu + k_1.k_0 ik_1^\mu \partial X^\mu$  and that of  $\lambda_{-2} = \frac{1}{6}\partial^2\rho$  is  $(\frac{1}{2}k_1.k_1 + 2k_2.k_0)$  as required.

### 2.1.4 Space-time Fields

We can define fields as usual [1, 2] by replacing  $k_1^\mu k_1^\nu$  by  $\Phi^{\mu\nu}$  and  $k_2^\mu$  by  $A^\mu$ . The gauge parameters are obtained by replacing  $\lambda_1 k_1^\mu$  by  $\epsilon^\mu$  and  $\lambda_2$  by  $\epsilon_2$ . Then we have the following:

**Constraints:** The mass shell constraint fixes  $p^2 + 2 = 0$ . In addition we have,

$$1. \quad p_\nu \Phi^{\nu\mu} + 2A^\mu = 0 \quad (2.1.17)$$

$$2. \quad \Phi_\nu^\nu + 4p_\nu A^\nu = 0 \quad (2.1.18)$$

**Gauge transformations:**

$$\begin{aligned} \delta\Phi^{\mu\nu} &= \eta^{\mu\nu} \epsilon_2 + p^{(\mu} \epsilon^{\nu)} \\ \delta A^\mu &= p^\mu \epsilon_2 + \epsilon^\mu \end{aligned} \quad (2.1.19)$$

Note that the constraints are not invariant under the gauge transformations unless  $\epsilon^\mu = \frac{3}{2}p^\mu \epsilon_2$ , along with the mass shell condition  $p^2 + 2 = 0$  and the critical dimension  $D = 26$ . These correspond to the zero norm states: states that are physical as well as pure gauge. It is easy to see that this gauge transformation corresponds to the state  $(L_{-2} + \frac{3}{2}L_{-1}^2)e^{ik_0 \cdot X}|0\rangle$ .

## 2.2 Level 3

### 2.2.1 Vertex Operators

The vertex operators in the OC formalism at this level can be written down as follows:

$$\begin{aligned} e^{ik_0 \cdot \tilde{Y} + ik_1 \cdot \tilde{Y}_1 + ik_2 \cdot \tilde{Y}_2 + ik_3 \cdot \tilde{Y}_3} |0\rangle = \\ (\dots + i \frac{k_3^\mu}{2!} \partial^3 X^\mu - k_2^\mu k_1^\nu \partial^2 X^\mu \partial X^\nu - i \frac{k_1^\mu k_1^\nu k_1^\rho}{3!} \partial X^\mu \partial X^\nu \partial X^\rho + \dots) e^{ik_0 \cdot X} |0\rangle \end{aligned} \quad (2.2.20)$$

### 2.2.2 Action of $L_{\pm n}$

Using the same equation (2.0.2) one gets:

$$\begin{aligned} \exp & [\lambda_3(ik_0.\tilde{Y}_3 - \tilde{Y}_1.\tilde{Y}_2) + \lambda_2(ik_1\tilde{Y}_3 + ik_0\tilde{Y}_2 - \frac{\tilde{Y}_1.\tilde{Y}_1}{2}) + \lambda_1(i2k_2\tilde{Y}_3 + ik_1\tilde{Y}_2 + ik_0\tilde{Y}_1) \\ & + \lambda_0(i3k_3\tilde{Y}_3 + 2ik_2\tilde{Y}_2 + ik_1\tilde{Y}_1 + \frac{k_0^2}{2}) + \lambda_{-1}(i3k_3\tilde{Y}_2 + i2k_2\tilde{Y}_1 + k_1.k_0) \\ & + \lambda_{-2}(i3k_3\tilde{Y}_1 + 2k_2.k_0 + \frac{k_1.k_1}{2}) + \lambda_{-3}(3k_3.k_0 + 2k_2.k_1)] e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2 + ik_3\tilde{Y}_3} |0\rangle \end{aligned} \quad (2.2.21)$$

One can extract as before the action of  $L_{+n}$  on the vertex operators by extracting terms involving  $k_3, k_2k_1$  and  $k_1k_1k_1$ . Similarly gauge transformations are obtained by extracting the level three terms.

1.  $\lambda_0 \mathbf{L}_0$  :

$$\lambda_0(3 + \frac{k_0^2}{2})[ik_3^\mu \tilde{Y}_3^\mu - k_2^\mu k_1^\nu \tilde{Y}_2^\mu \tilde{Y}_1^\nu - ik_1^\mu k_1^\nu k_1^\rho \tilde{Y}_1^\mu \tilde{Y}_1^\nu \tilde{Y}_1^\rho] \quad (2.2.22)$$

2.  $\lambda_{-1} \mathbf{L}_1$  :

$$\lambda_{-1}[(i3k_3^\mu + ik_2^\mu k_1.k_0)\tilde{Y}_2^\mu - (2k_2^\mu k_1^\nu + \frac{k_1^\mu k_1^\nu}{2}k_1.k_0)\tilde{Y}_1^\mu \tilde{Y}_1^\nu] \quad (2.2.23)$$

3.  $\lambda_{-2} \mathbf{L}_2$  :

$$\lambda_{-2}[i3k_3^\mu \tilde{Y}_1^\mu + 2k_2.k_0 ik_1^\mu \tilde{Y}_1^\mu + \frac{k_1.k_1}{2} ik_1^\mu \tilde{Y}_1^\mu] \quad (2.2.24)$$

4.  $\lambda_1 \mathbf{L}_{-1}$  :

$$\lambda_1[i2k_2^\mu \tilde{Y}_3^\mu - k_1^\mu k_1^\nu \tilde{Y}_2^\mu \tilde{Y}_1^\nu - k_0^\mu k_2^\nu \tilde{Y}_1^\mu \tilde{Y}_2^\nu] \quad (2.2.25)$$

5.  $\lambda_2 \mathbf{L}_{-2}$  :

$$\lambda_2[ik_1^\mu \tilde{Y}_3^\mu - k_0^\mu k_1^\nu \tilde{Y}_2^\mu \tilde{Y}_1^\nu - ik_1^\mu \tilde{Y}_1^\mu \frac{\tilde{Y}_1.\tilde{Y}_1}{2}] \quad (2.2.26)$$

6.  $\lambda_3 \mathbf{L}_{-3}$  :

$$\lambda_3[ik_0^\mu \tilde{Y}_3^\mu - \tilde{Y}_1.\tilde{Y}_2] \quad (2.2.27)$$

### 2.2.3 Liouville Mode

Exactly as in the level two case one can get the Liouville mode dependences - both the classical and anomalous terms.

$$e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2+ik_3.\tilde{Y}_3} =: e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2+ik_3.\tilde{Y}_3} ;$$

$$e^{\frac{1}{2}[k_0^2\langle XX\rangle+2k_1.k_0\langle X\partial X\rangle+k_1.k_1\langle\partial X\partial X\rangle+2k_2.k_0\langle X\partial^2 X\rangle+2k_3.k_0\langle\frac{\partial^3 X}{2!}X\rangle+2k_2.k_1\langle\partial^2 X\partial X\rangle]}$$

This is the anomalous dependence. Using covariant derivatives gives the classical part also:

$$=: e^{ik_0.X+ik_1.e^\rho\partial X+ik_2.e^{2\rho}(\partial^2 X+\partial\rho\partial X)+ik_3.e^{3\rho}(\partial^3 X+3\partial^2 X\partial\rho+\partial X\partial^2\rho)} ;$$

$$e^{\frac{1}{2}[k_0^2\rho+2k_1.k_0\frac{1}{2}\partial\rho+k_1.k_1\frac{1}{6}\partial^2\rho+2k_2.k_0\frac{1}{3}\partial^2\rho+2k_3.k_0\frac{\partial^3\rho}{8}+2k_2.k_1\frac{\partial^3\rho}{12}]}$$
 (2.2.28)

We have used  $\langle\partial^3 XX\rangle = \frac{\partial^3\rho}{4}$  and  $\langle\partial^2 X\partial X\rangle = \frac{\partial^3\rho}{12}$ . Using the connection between  $\lambda_n$  and  $\partial^n\rho$  one can check that this is the same as the results of section 2.2.2. In this form it is easier to compare with LV formalism.

### 2.2.4 Space-time Fields

We introduce space-time fields as before by replacing  $k_1^\mu k_1^\nu k_1^\rho$  by  $\Phi^{\mu\nu\rho}$ ,  $k_2^\mu k_1^\nu$  by  $B^{\mu\nu} + C^{\mu\nu}$  where  $B$  is symmetric and  $C$  is antisymmetric, and  $k_3^\mu$  by  $A^\mu$ . For the gauge parameters we let  $\lambda_3$  be  $\epsilon_3$ ,  $\lambda_2 k_1^\mu$  be  $\epsilon_{12}^\mu$ ,  $\lambda_1 k_2^\mu$  be  $\epsilon_{21}^\mu$  and  $\lambda_1 k_1^\mu k_1^\nu$  be  $\epsilon_{111}^{\mu\nu}$ . We then have:

#### Constraints:

The mass shell constraint  $L_0 = 1$  becomes  $p^2 + 4 = 0$ . In addition,

1.

$$p_\nu(B^{\mu\nu} + C^{\mu\nu}) + 3A^\mu = 0$$
 (2.2.29)

2.

$$\frac{p_\rho\Phi^{\mu\nu\rho}}{2} + B^{\mu\nu} = 0$$
 (2.2.30)

3.

$$\frac{B_\nu^\nu}{12} + \frac{p_\nu A^\nu}{8} = 0$$
 (2.2.31)

4.

$$\frac{\Phi_\nu^{\mu\nu}}{12} + \frac{A^\mu}{2} + \frac{p_\nu B^{\nu\mu}}{3} = 0$$
 (2.2.32)

### Gauge Transformations:

$$\begin{aligned}
\delta\Phi^{\mu\nu\rho} &= \epsilon_{12}^{(\mu}\eta^{\nu\rho)} + p^{(\mu}\epsilon_{111}^{\nu\rho)} \\
\delta(B^{\mu\nu} + C^{\mu\nu}) &= p^\mu\epsilon_{12}^\nu + p^\nu\epsilon_{21}^\mu + \epsilon_{111}^{\mu\nu} + \epsilon_3\eta^{\mu\nu} \\
\delta A^\mu &= p^\mu\epsilon_3 + \epsilon_{12}^\mu + 2\epsilon_{21}^\mu
\end{aligned} \tag{2.2.33}$$

(A symmetrization has been indicated in the first line, which involves adding two other orderings giving three permutations for each term.)

## 3 Loop Variable Formalism

The loop variable formalism is gauge invariant. This requires auxiliary fields that are obtained by introducing an extra coordinate that we will call  $\theta$  and it's conjugate momentum  $q$ . This can be thought of as a dimensional reduction process in a  $D+1$  dimensional theory. The zero mode  $q_0$  becomes the mass of the field. However it is important to note that the spectrum of string theory requires that it is  $q_0^2$  that has to be integers, and not  $q_0$  as in usual Kaluza Klein dimensional reduction.

We begin with the following loop variable and define covariantized vertex operators from it [1, 2]:

$$e^{i\int_c \alpha(s)k(s)\partial_z X(z+s)ds + ik_0 X(z)} = e^{i\sum_{n\geq 0} k_n Y_n(z)} \tag{3.0.34}$$

where

$$\begin{aligned}
Y_n &= \frac{\partial Y}{\partial x_n}, \quad n > 0, \quad Y_0 \equiv Y \\
Y &= \sum_{n\geq 0} \alpha_n \tilde{Y}_n
\end{aligned} \tag{3.0.35}$$

and  $\alpha_n$  are given by  $\alpha(s) = \sum_{n\geq 0} \alpha_n s^{-n} = e^{\sum_{n\geq 0} x_n s^{-n}}$  with  $\alpha_0 = x_0 \equiv 1$ . The following relation is useful:  $\frac{\partial \alpha_n}{\partial x_m} = \alpha_{n-m}$ .

The vertex operator in (3.0.34) is covariantized in the sense that there is a well defined action of a gauge transformation on it:  $k(s) \rightarrow k(s)\lambda(s)$  and equivalently, if we define  $\lambda(s) = \sum_{n\geq 0} \lambda_n s^{-n}$  (with  $\lambda_0 \equiv 1$ ), the gauge transformation is:  $k_n \rightarrow k_n + \lambda_p k_{n-p}$ .

The equations obtained from this are automatically invariant under this transformation because the prescription is to integrate over  $\alpha(s)$ . Thus a

multiplication by  $\lambda(s)$  does not affect the final result since it can be re-absorbed into  $\alpha(s)$ . The equations are obtained by demanding Weyl invariance of the loop variable with Liouville mode dependence.

The Liouville mode dependence is obtained using

$$\langle YY \rangle \equiv \Sigma, \quad \langle Y_n Y \rangle = \frac{1}{2} \frac{\partial \Sigma}{\partial x_n}, \quad \langle Y_n Y_m \rangle = \frac{1}{2} \left( \frac{\partial^2 Y}{\partial x_n \partial x_m} - \frac{\partial Y}{\partial x_{n+m}} \right) \quad (3.0.36)$$

Here  $\Sigma$  (defined by the first equation) is a covariantized version of the Liouville mode  $\rho$  and is a linear combination of  $\rho$  and its derivatives. As explained in the last section there is also a classical Liouville mode dependence that one needs to include. In the present formalism this need not be included. They are obtained by identifying some of the auxiliary fields that are present with the physical fields. Thus *all* the Liouville mode dependence comes from anomalies. This will become clear in the examples below. The loop variable with its  $\Sigma$  dependence is thus:

$$: e^{i \sum_{n \geq 0} k_n Y_n(z)} : e^{\frac{1}{2} [k_0 \cdot k_0 \Sigma + k_n \cdot k_0 (\frac{\partial \Sigma}{\partial x_n}) + \frac{1}{2} \sum_{n,m > 0} k_n \cdot k_m (\frac{\partial^2 \Sigma}{\partial x_n \partial x_m} - \frac{\partial \Sigma}{\partial x_{n+m}})]} \quad (3.0.37)$$

We consider the level two and three operators in turn.

### 3.1 Level 2

#### 3.1.1 Vertex Operators

$$\begin{aligned} & e^{\frac{(k_0^2 + q_0^2)}{2} \Sigma} [i k_2^\mu Y_2^\mu + i q_2 \theta_2 - \frac{k_1^\mu k_1^\nu}{2} Y_1^\mu Y_1^\nu - \frac{q_1 q_1}{2} \theta_1 \theta_1 - k_1^\mu q_1 Y_1^\mu \theta_1 \\ & + i(k_1^\mu Y_1^\mu + q_1 \theta_1)(k_1 \cdot k_0 + q_1 q_0) \frac{1}{2} \frac{\partial \Sigma}{\partial x_1} + (k_2 \cdot k_0 + q_2 q_0) \frac{1}{2} \frac{\partial \Sigma}{\partial x_2} + \\ & - i \frac{k_1^\mu k_1^\nu}{2} q_1 Y_1^\mu Y_1^\nu \theta_1 - i \frac{q_1 q_1}{2} k_1^\mu \theta_1 Y_1^\mu \\ & \frac{(k_1 \cdot k_1 + q_1 q_1)}{2} \frac{1}{2} (\frac{\partial^2 \Sigma}{\partial x_1^2} - \frac{\partial \Sigma}{\partial x_2})] e^{i k_0 \cdot Y} \end{aligned} \quad (3.1.38)$$

Weyl Invariance is independence of  $\Sigma$ . The coefficients of  $\Sigma$  and its derivatives have to be set to zero. There are the constraints. It will be seen that field redefinitions will make these equivalent to the constraints of the OC formalism (2.1.17) and (2.1.18). This implies that the classical Liouville mode dependence is included here indirectly through terms involving  $q_n$ .

### 3.1.2 Space-time Fields, Gauge Transformations and Constraints

- **Space-time Fields:**

The fields are obtained by setting  $k_1^\mu k_1^\nu \approx S_{11}^{\mu\nu}$ ,  $k_2^\mu \approx S_2^\mu$ ,  $k_1^\mu q_1 q_0 \approx S_{11}^\mu$ ,  $q_1 q_1 \approx S_{11}$ , and  $q_2 q_0 \approx S_2$ . The gauge parameters are  $\lambda_2 \approx \Lambda_2$ ,  $\lambda_1 k_1^\mu \approx \Lambda_{11}^\mu$ ,  $\lambda_1 q_1 q_0 \approx \Lambda_{11}$ .

- **Gauge Transformations:**

$$\begin{aligned} \delta S_{11}^{\mu\nu} &= p^{(\mu} \Lambda_{11}^{\nu)}, \quad \delta S_2^\mu = \Lambda_{11}^\mu + p^\mu \Lambda_2, \quad \delta S_{11}^\mu = \Lambda_{11}^\mu + p^\mu \Lambda_{11}, \\ \delta S_2 &= \Lambda_2 q_0^2 + \Lambda_{11}, \quad \delta S_{11} = 2\Lambda_{11} \end{aligned} \quad (3.1.39)$$

Now one can make the following identifications:  $q_1 q_1 \approx q_2 q_0$ ,  $k_1^\mu q_1 \approx k_2^\mu q_0$ , and  $\lambda_1 q_1 \approx \lambda_2 q_0$ . This gives:  $S_{11}^\mu = 2S_2^\mu$ ,  $S_{11} \approx S_2$  and  $\Lambda_{11} \approx 2\Lambda_2$  and the gauge transformations are consistent with these identifications.

- **Constraints:**

The coefficient of  $\Sigma$  gives the usual mass shell condition  $p^2 + q_0^2 = 0$ . Note that the  $(mass)^2$  equals the dimension of the operator, but the  $\Sigma$  dependence representing this (and also all other  $\Sigma$  dependences) comes from an anomaly rather than from the classical dependence as in the OC formalism.

1. Coefficient of  $\frac{\partial^2 \Sigma}{\partial x_1^2}$

$$k_1 \cdot k_1 + q_1 q_1 = 0 \quad \Rightarrow \quad S_{11\mu}^\mu + S_2 = 0 \quad (3.1.40)$$

2. Coefficient of  $\frac{\partial \Sigma}{\partial x_2}$

$$k_2 \cdot k_0 + q_2 q_0 = 0 \quad \Rightarrow \quad p_\mu S_2^\mu + S_2 = 0 \quad (3.1.41)$$

3. Coefficient of  $\frac{\partial \Sigma}{\partial x_1} Y_1^\mu$

$$(k_1 \cdot k_0 + q_1 q_0) k_1^\mu = 0 \quad \Rightarrow \quad p_\nu S_{11}^{\mu\nu} + 2S_2^\mu = 0 \quad (3.1.42)$$

The constraint proportional to  $\theta_1$  is seen to be a linear combination of the above. The equations of motion are obtained by setting the variational derivative of  $\Sigma$  equal to zero, and are gauge invariant.

## 3.2 Level 3

### 3.2.1 Vertex Operator

The complete Level 3 gauge covariantized vertex operator is:

$$\begin{aligned}
& e^{\frac{(k_0^2 + q_0^2)}{2}\Sigma} [ik_3^\mu Y_3^\mu + iq_3\theta_3 - k_2^\mu k_1^\nu Y_2^\mu Y_1^\nu - \\
& q_1 q_2 \theta_1 \theta_2 - k_1^\mu q_2 Y_1^\mu \theta_2 - k_2^\mu q_1 Y_2^\mu \theta_1 - i \frac{k_1^\mu k_1^\nu k_1^\rho}{3!} Y_1^\mu Y_1^\nu Y_1^\rho - i \frac{(q_1)^3}{3!} (\theta_1)^3 \\
& + i(k_2^\mu Y_2^\mu + q_2 \theta_2)(k_1 \cdot k_0 + q_1 q_0) \frac{1}{2} \frac{\partial \Sigma}{\partial x_1} \\
& + i(k_1^\mu Y_1^\mu + q_1 \theta_1) [(k_2 \cdot k_0 + q_2 q_0) \frac{1}{2} \frac{\partial \Sigma}{\partial x_2} + \frac{1}{2} (k_1 \cdot k_1 + q_1 q_1) \frac{1}{2} (\frac{\partial^2 \Sigma}{\partial x_1 \partial x_1} - \frac{\partial \Sigma}{\partial x_2})] \\
& + (k_3 \cdot k_0 + q_3 q_0) \frac{1}{2} \frac{\partial \Sigma}{\partial x_3} + (k_2 \cdot k_1 + q_2 q_1) \frac{1}{2} (\frac{\partial^2 \Sigma}{\partial x_1 \partial x_2} - \frac{\partial \Sigma}{\partial x_3})] e^{ik_0 \cdot Y} \quad (3.2.43)
\end{aligned}$$

### 3.2.2 Space-time Fields, Gauge Transformations and Constraints

- Space-time Fields

$$\begin{aligned}
& k_1^\mu k_1^\nu k_1^\rho \approx S_{111}^{\mu\nu\rho}, \quad k_1^\mu k_1^\nu q_1 q_0 \approx S_{111}^{\mu\nu}, \quad k_2^\mu k_1^\nu \approx S_{21}^{\mu\nu} \\
& k_1^\mu q_1 q_1 \approx S_{111}^\mu, \quad k_1^\mu q_2 q_0 \approx S_{12}^\mu, \quad k_2^\mu q_1 q_0 \approx S_{21}^\mu, \quad k_3^\mu \approx S_3^\mu \\
& q_3 q_0 \approx S_3, \quad q_2 q_1 \approx S_{21}, \quad q_1 q_1 q_1 q_0 \approx S_{111} \quad (3.2.44)
\end{aligned}$$

- Gauge Parameters

$$\begin{aligned}
& \lambda_1 k_1^\mu k_1^\nu \approx \Lambda_{111}^{\mu\nu}, \quad \lambda_2 k_1^\mu \approx \Lambda_{12}^\mu, \\
& \lambda_1 k_2^\mu \approx \Lambda_{21}^\mu, \quad \lambda_1 k_1^\mu q_1 q_0 \approx \Lambda_{111}^\mu \\
& \lambda_3 \approx \Lambda_3, \quad \lambda_2 q_1 q_0 \approx \Lambda_{12}, \quad \lambda_1 q_1 q_1 \approx \Lambda_{111}, \quad \lambda_1 q_2 q_0 \approx \Lambda_{21} \quad (3.2.45)
\end{aligned}$$

- Gauge Transformations

$$\begin{aligned}
\delta S_{111}^{\mu\nu\rho} &= p^{(\mu} \Lambda_{111}^{\nu\rho)} \\
\delta S_{111}^{\mu\nu} &= 4\Lambda_{111}^{\mu\nu} + p^{(\mu} \Lambda_{111}^{\nu)} \\
\delta S_{21}^{\mu\nu} &= \Lambda_{111}^{\mu\nu} + p^\mu \Lambda_{12}^\nu + p^\nu \Lambda_{21}^\mu \\
\delta S_{111}^\mu &= 2\Lambda_{111}^\mu + p^\mu \Lambda_{111}
\end{aligned}$$

$$\begin{aligned}
\delta S_{12}^\mu &= \Lambda_{111}^\mu + 4\Lambda_{12}^\mu + p^\mu \Lambda_{21} \\
\delta S_{21}^\mu &= \Lambda_{111}^\mu + 4\Lambda_{21}^\mu + p^\mu \Lambda_{12} \\
\delta S_3^\mu &= \Lambda_{21}^\mu + \Lambda_{12}^\mu + p^\mu \Lambda_3 \\
\delta S_{111} &= 12\Lambda_{111} \\
\delta S_{21} &= \Lambda_{21} + \Lambda_{111} + \Lambda_{12} \\
\delta S_3 &= \Lambda_{21} + \Lambda_{12} + 4\Lambda_3
\end{aligned} \tag{3.2.46}$$

• **Constraints**

$$1. \quad k_3.k_0 + q_3q_0 = 0 \Rightarrow p_\nu S_3^\nu + S_3 = 0 \tag{3.2.47}$$

$$2. \quad k_2.k_1 + q_2q_1 = 0 \Rightarrow S_{21\mu}^\mu + S_{21} = 0 \tag{3.2.48}$$

$$3. \quad (k_1.k_1 + q_1q_1)k_1^\mu = 0 \Rightarrow S_{111\nu}^{\nu\mu} + S_{111}^\mu = 0 \tag{3.2.49}$$

$$4. \quad (k_2.k_0 + q_2q_0)k_1^\mu = 0 \Rightarrow p_\nu S_{21}^{\nu\mu} + S_{12}^\mu = 0 \tag{3.2.50}$$

$$5. \quad (k_1.k_0 + q_1q_0)k_1^\mu k_1^\nu = 0 \Rightarrow p_\rho S_{111}^{\mu\nu\rho} + S_{111}^{\mu\nu} = 0 \tag{3.2.51}$$

$$6. \quad (k_1.k_0 + q_1.q_0)k_2^\mu = 0 \Rightarrow p_\nu S_{21}^{\mu\nu} + S_{21}^\mu = 0 \tag{3.2.52}$$

$$7. \quad (k_2.k_0 + q_2q_0)q_1q_0 = 0 \Rightarrow p_\nu S_{21}^\nu + 4S_{21} = 0 \tag{3.2.53}$$

Note that the last constraint comes from  $\frac{\partial \Sigma}{\partial x_2} \theta_1$ .

## 4 Mapping from OC Formalism to LV Formalism

### 4.1 Level 2

#### 4.1.1 Mapping of fields

The mapping is given by:

$$\begin{aligned}
\Phi^{\mu\nu} &= S_{11}^{\mu\nu} + (\eta^{\mu\nu} + \frac{5}{2}p^\mu p^\nu)S_2 \\
A^\mu &= S_2^\mu + 2p^\mu S_2
\end{aligned} \tag{4.1.54}$$

### 4.1.2 Mapping Gauge Transformations

If one makes a LV gauge transformation with parameters  $\Lambda_{11}^\mu$  and  $\Lambda_2$  one obtains:

$$\begin{aligned}\delta\Phi^{\mu\nu} &= p^{(\mu}\Lambda_{11}^{\nu)} + (\eta^{\mu\nu} + \frac{5}{2}p^\mu p^\nu)4\Lambda_2 \\ \delta A^\mu &= \Lambda_{11}^\mu + 9 p^\mu \Lambda_2\end{aligned}\tag{4.1.55}$$

The relative values of the different terms in (4.1.54), and therefore in the gauge transformation (4.1.55), are fixed by requiring that the gauge transformation be generated by some combination of  $L_{-n}$ 's. One can check that (4.1.55) corresponds to a gauge transformation by  $4\Lambda_2(L_{-2} + \frac{5}{4}L_{-1}^2) + \Lambda_{11}^\mu L_{-1}$ .

### 4.1.3 Mapping Constraints

Now consider the constraint (2.1.18). We see that

$$p_\nu \Phi^{\nu\mu} + 2A^\mu = p_\nu S_{11}^{\mu\nu} + 2S_2^\mu + (5 - \frac{5}{2}q_0^2)p^\mu S_2\tag{4.1.56}$$

Only for  $q_0^2 = 2$  does it become the LV constraint  $p_\nu S_{11}^{\mu\nu} + 2S_2^\mu$ . Furthermore

$$4p.A + \Phi_\mu^\mu = 4p.S_2 + S_{11\mu}^\mu + [(D - \frac{5}{2}q_0^2) - 8q_0^2]S_2\tag{4.1.57}$$

This should equal

$$4(p.S_2 + S_2) + S_{11\mu}^\mu + S_2\tag{4.1.58}$$

This fixes  $D = 26$  (using  $q_0^2 = 2$ ). Thus we see that while the LV equations are gauge invariant in any dimension, when we require equivalence with OC formalism the critical dimension is picked out.

### 4.1.4 Equivalence of OC and LV Formalisms

Now we can see that the LV formalism is equivalent to the OC formalism: Start with a vertex operator in the OC formalism with fields that obey (4.1.56, 4.1.57). This implies that the corresponding LV vertex operator obeys the same constraint (4.1.56). Similarly (4.1.57) implies (4.1.58). This is the sum of two constraints of the LV formalism. Since the LV formalism is gauge invariant one can choose a gauge (using invariance under  $\Lambda_2$  transformations), where  $S_2$  to equal  $-p.S_2$ . This implies (by the constraint)

that  $S_{11\mu}^\mu + S_2 = 0$ . Thus if the fields obey the physical state constraints of the OC formalism, then using the gauge invariance, we see that the LV constraints are also satisfied. In the reverse direction it is easier because we just have to take a linear combination of two LV constraints (3.1.40-3.1.42) to get an OC constraint.

We can go further in analyzing the constraints. After obtaining  $p.S_2 + S_2 = 0$  using a  $\Lambda_2$  transformations, there is a further invariance involving both  $\Lambda_{11}^\mu$  and  $\Lambda_2$  with  $p.\Lambda_{11} + q_0^2 \Lambda_2 = 0$ . This transformation preserves all the constraints. (We also have to use the mass shell condition  $p^2 + q_0^2 = 0$ .) Using this invariance we can set  $S_2 = 0$  while preserving  $p.S_2 + S_2 = 0 = S_{11\mu}^\mu + S_2$ . This then implies that  $p.S_2 = S_{11\mu}^\mu = 0$ . A very similar analysis done on the OC side using the constraint  $4p.A + \Phi_\mu^\mu = 0$  and the gauge transformations with  $\epsilon^\mu + \frac{3}{2}p^\mu\epsilon$  that preserves the constraint (provided  $D = 26$ ,  $q_0^2 = 2$ ): we can use it to set  $p.A$  to zero and so  $\Phi_\mu^\mu = 0$ . Thus on both sides we have a transverse vector and a traceless tensor obeying  $p_\nu\Phi^{\nu\mu} + 2A^\mu = 0$ .

There are also terms involving  $\theta_n$  on the LV side. But if we focus on the equations of motion involving only vertex operators with  $Y_n$ , then the two systems are identical. Since the physical states of the string are conjugate to these vertex operators (that have only  $Y_n$ ), this is all we need to describe the physics of string theory.

Finally we can use transverse gauge transformations involving  $\epsilon^\mu$  (i.e with  $p.\epsilon = 0$ ) to gauge away the transverse vector  $A^\mu$  (and the same thing can be done on the LV side to gauge away  $S_2^\mu$ ). This leaves a tensor  $\Phi^{\mu\nu}$  which is transverse -  $p_\nu\Phi^{\nu\mu} = 0$  - and traceless. This is the right number of degrees of freedom for the first massive state of the bosonic open string.

This concludes the demonstration of the equivalence of the OC formalism and LV formalism for Level 2 at the free level. In the next subsection we discuss the Level 3 system.

## 4.2 Level 3

There are four constraints on the OC side that need to be mapped to the LV side. This involves mapping the fields as well as gauge transformation parameters.  $q_0^2 = 4$  is obviously required from the mass shell condition. Furthermore mapping all the constraints is possible only when  $D = 26$ . The result is the following:

### 4.2.1 Mapping Fields

$$\begin{aligned}
S_{111}^{\mu\nu\rho} + S_{\beta}^{(\mu}\eta^{\nu\rho)} &= \Phi^{\mu\nu\rho} \\
\frac{S_{111}^{\mu\nu}}{4} + \eta^{\mu\nu}S_{\alpha} - \frac{1}{4}p^{(\mu}S_{\beta}^{\nu)} &= B^{\mu\nu} \\
S_{21}^{\mu\nu} + \eta^{\mu\nu}S_{\alpha} &= B^{\mu\nu} + C^{\mu\nu} \\
S_{21}^{\mu} - p^{\mu}S_{\alpha} &= 3A^{\mu}
\end{aligned} \tag{4.2.59}$$

Here  $S_{\beta}^{\mu}$  is defined by the requirement  $\delta S_{\beta}^{\mu} = \Lambda_{12}^{\mu}$  and  $S_{\alpha}$  is defined by  $\delta S_{\alpha} = \epsilon^3$ . Also

$$\begin{aligned}
S_{111}^{\mu} &= -\frac{1}{2}S_{12}^{\mu} - \frac{3}{2}S_{21}^{\mu} + 14S_3^{\mu} - 14p^{\mu}S_{\beta} + 2p^{\mu}S_{\alpha} \\
S_{\beta}^{\mu} &= \frac{1}{2}\left[\frac{S_{12}^{\mu} - S_{21}^{\mu}}{4} + S_3^{\mu} - p^{\mu}S_{\beta}\right] \\
S_{\beta} &= \frac{S_3}{4} - 2S_{\alpha} \\
S_{21} &= \frac{28}{3}S_{\alpha}
\end{aligned} \tag{4.2.60}$$

The gauge parameters obey:

$$\begin{aligned}
\Lambda_{12} &= 4\epsilon^3 \\
\Lambda_{111}^{\mu\nu} &= \epsilon_{111}^{\mu\nu} \\
\Lambda_{12}^{\mu} &= \epsilon_{12}^{\mu} \\
\Lambda_{21}^{\mu} &= \epsilon_{21}^{\mu} \\
\Lambda_{111}^{\mu} &= 3\epsilon_{12}^{\mu} + 2\epsilon_{21}^{\mu} \\
\Lambda_{111} &= \frac{4}{3}\Lambda_{12} - \Lambda_{21} \\
\Lambda_{21} &= \frac{7}{18}\Lambda_{12}
\end{aligned} \tag{4.2.61}$$

Thus after imposing these relations we have one scalar parameter and two vector parameters and one tensor parameters. This is just enough to compare with the OC formalism. Let us see how the states of the bosonic string are described: One can truncate the theory by setting some gauge invariant combinations of fields to zero. The gauge invariant theory has two vectors, which can be taken to be  $S_3^{\mu}, S_{12}^{\mu}$ . One can truncate the theory so that the other vector  $S_{21}^{\mu}$  can be expressed in terms of these two, and we

already have a relation for  $S_{111}^\mu$  in terms of these three in the above equations. The truncation preserves the gauge invariance. These two remaining vectors,  $S_3^\mu, S_{12}^\mu$  can further be *gauged* to zero to get a gauge fixed theory. Also the traceless part of  $S_{111}^{\mu\nu}$  can be gauged to zero using  $\Lambda_{111}^{\mu\nu}$ . The constraint  $k_2.k_1 + q_2q_1 = 0$  implies that  $\lambda_1(k_1.k_1 + q_1q_1) = 0$ . Thus the trace of the tensor is related to the scalar by this constraint. Thus we are left with one scalar and one transverse three tensor  $S_{111}^{\mu\nu\rho}$ . The three tensor can be decomposed into a traceless part and a trace, which is a vector. This is the right degrees for a covariant description of a traceless transverse three index tensor [12, 13, 14]. There is also one two index anti symmetric tensor. These are the correct states for the third massive level of a bosonic open string.

Let us see how the constraints are mapped: The constraint  $p_\nu(B^{\mu\nu} + C^{\mu\nu}) + 3A^\mu = 0$  becomes (6):  $p_\nu S_{21}^{\mu\nu} + S_{21}^\mu = 0$ . (If we further choose a gauge where the scalar is set to zero - we have already truncated the theory so that there is only one scalar - then  $S_\alpha = 0$  and  $S_{21}^\mu = 3A^\mu$ .)

The second constraint  $\frac{p_\rho \Phi^{\mu\nu\rho}}{4} + B^{\mu\nu} = 0$  becomes a linear combination of (1) and (5):  $p_\rho S_{111}^{\mu\nu\rho} + S_{111}^{\mu\nu\rho} + \eta^{\mu\nu} \frac{(p.S_3 + S_3)}{8} = 0$ . Use a gauge transformation to choose the scalar field such that  $p.S_3 + S_3 = 0$ . Thus both constraints (1) and (5) are satisfied.

The constraint  $\frac{B_\nu^\nu}{12} + \frac{p_\nu A^\nu}{8} = 0$  becomes (2) + (7). One can use  $p.\Lambda_{21}$  to gauge (7) to zero, so that (2) is also satisfied.

The remaining constraint is a linear combination of (3) and (4). One can use  $\Lambda_{12}^\mu$  to set (4) to zero and therefore (3) also.

Thus we have seen that the OC constraints imply the LV constraints. The reverse is obvious. Having set one vector to zero and the scalar being constrained to equal the trace of the tensor, we also see that the field content is the same.

This proves that the vertex operators are exactly the same in both formalisms.

In the next section we see how this can be generalized to the interacting theory.

## 5 Interacting Theory

One of the interesting features of the LV formalism is that the interacting theory looks very similar to the free theory. In fact if one replaces

$$k^\mu(s) \rightarrow \int_0^R dt \bar{k}^\mu(s, t), \quad \Sigma \rightarrow \Sigma + G \quad (5.0.62)$$

where  $G$  is the regulated Green function, one gets the interacting theory. The LV momentum  $\bar{k}^\mu(s, t)$  is defined in terms of it's modes as follows:

$$\begin{aligned} \bar{k}_1^\mu(t) &= k_1^\mu(t) + tk_0^\mu(t) \\ \bar{k}_2^\mu(t) &= k_2^\mu(t) + tk_1^\mu(t) + \frac{t^2}{2}k_0^\mu(t) \\ &\vdots \\ \bar{k}_n^\mu(t) &= \sum_0^n k_m^\mu(t) D_m^n t^{n-m} \\ \bar{k}_n^\mu(t) &= k_n^\mu(t) + (n-1)tk_{n-1}^\mu(t) + \dots + \frac{t^n}{n}k_0^\mu(t) \end{aligned} \quad (5.0.63)$$

where

$$\begin{aligned} D_m^n &= {}^{n-1}C_{m-1}, \quad n \geq m > 0 \\ D_0^n &= \frac{1}{n}, \quad n \neq 0 \end{aligned} \quad (5.0.64)$$

This implements the following: In the interacting theory there are many vertex operators defined by their location  $t$  on the world sheet boundary. As an example we have  $k_1^\mu(t)\partial X^\mu(t)e^{ik_0 \cdot X(t)}$ . We do a Taylor expansion of these vertex operators about the point 0. Thus

$$\begin{aligned} e^{ik_0 \cdot X(t) + ik_1 \cdot \partial X(t)} = \\ e^{ik_0 \cdot [X(0) + t\partial X(0) + \frac{t^2}{2!}\partial^2 X(0) + \dots] + ik_1(t)[\partial X(0) + t\partial^2 X(0) + \frac{1}{2!}\partial^3 X(0) + \dots]} \end{aligned} \quad (5.0.65)$$

This Taylor expansion brings all the vertex operators to one point and it becomes one generalized vertex operator of the form  $e^{i\sum_{n \geq 0} k_n \tilde{Y}_n(z)}$  with the more complicated  $k_n$  given in (5.0.63). This operation is well defined because we have a world sheet cutoff that regulates the theory and coincident operators are well defined. Once we have done this, we have something that, mathematically, looks like a single vertex operator. In correlation functions this is equivalent to Taylor expanding the Green function  $G(0, t) = G(0, 0) +$

$t\partial G(0,0) + \dots$ . This single vertex operator can then be covariantized using the  $\alpha_n$  as described in Section 2 and gauge invariant equations obtained. The gauge transformations are exactly the same as in the free case, except for the replacement of  $k_n$  by  $\int_0^R dt \bar{k}_n(t)$  and  $\lambda_n$  by  $\int_0^R dt \lambda(t)$ . There is one technical point:  $\bar{q}_n(t) = q_n(t)$  and one can assume that  $G^{\theta\theta}(0,t) = 0$  - this is necessary in order to reproduce string correlation functions [2].

Once we have this structure we can implement the results of the previous section: Represent the product of vertex operators in the OC formalism as well as in the LV formalism as one vertex operator using the above construction. Take the Level 2 map from OC to LV (4.1.54) of the free theory and apply it to the Level 2 of this interacting theory. In order to do this let us first rewrite the mapping in terms of the loop variable momenta. Let  $K$  denote the OC variables in the representation (2.0.1) and  $k, q$  denote the LV variables. The constraint (2.1.7) reads

$$[2k_2.k_0 + \frac{1}{2}k_1.k_1] = 0$$

This has to be mapped to the linear combination of two LV constraints-

$$[k_2.k_0 + q_2q_0] + \frac{1}{2}[k_1.k_1 + q_1q_1] = 0$$

(We also have the identification  $q_1q_1 = q_2q_0$ .) This is done by means of the mapping (4.1.54). Expressed in terms of  $k, q, K$  the map reads:

$$\begin{aligned} K_1^\mu K_1^\nu &= k_1^\mu k_1^\nu + (\eta^{\mu\nu} + \frac{5}{2}k_0^\mu k_0^\nu)q_2q_0 \\ K_2^\mu &= k_2^\mu + 2k_0^\mu q_2q_0 \end{aligned} \tag{5.0.66}$$

It is easy to generalize this to the interacting case using the method described above:

$$\begin{aligned} \int_0^1 dt \int_0^1 dt' \bar{K}_1^\mu(t) \bar{K}_1^\nu(t') &= \int_0^1 dt_1 \int_0^1 dt_2 [\bar{k}_1^\mu(t_1) \bar{k}_1^\nu(t_2) \\ &\quad + (\eta^{\mu\nu} + \frac{5}{2} \int_0^1 dt_3 \int_0^1 dt_4 k_0^\mu(t_3) k_0^\nu(t_4)) q_2(t_1) q_0(t_2)] \\ \int_0^1 dt \bar{K}_2^\mu(t) &= \int_0^1 dt \bar{k}_2^\mu(t) + 2 \int_0^1 dt_3 k_0^\mu(t_3) \int_0^1 dt_1 \int_0^1 dt_2 q_2(t_1) q_0(t_2) \end{aligned} \tag{5.0.67}$$

Note that we have set  $R = 1$ .<sup>1</sup> As in all such cases of relations between loop variable momenta, the mapping to space-time fields has to be done in a recursive way. In each equation the highest level field gets defined in terms of lower level fields. Thus the map between  $\Phi^{\mu\nu}$  and  $S_{11}^{\mu\nu}$  is defined once the map between  $A^\mu$  ( $\approx K_1^\mu$ ) and  $k_1^\mu, q_1$  ( $\approx S_1^\mu, S_1$ ) are known from a lower level equation. This becomes one of the inputs at the next stage, which is the map between  $\Phi^{\mu\nu\rho}$  and  $S_{111}^{\mu\nu\rho}$ , and so on.

Once we have made the map, one can generalize the earlier sequence of arguments to establish the equivalence of the S-matrices modulo the following assumption - *we assume that the equivalence between the free OC formalism and free LV formalism exists at all levels in the same way as was shown explicitly for Level 2 and 3.* This is a very plausible assumption because the free theory is determined by the field content - which is the same for LV formalism and BRST string theory. The gauge transformation is also the same modulo field redefinitions. Therefore in the critical dimension (which is when BRST string theory is gauge invariant) one expects the two theories to be completely equivalent at the free level. The sequence of arguments for the equivalence of the formalisms in the interacting case is:

1. We assume that the external fields of the OC string obey physical state conditions. Then the  $\rho$  dependence drops out from the vertex operator. The dependence on the constant part of  $\rho$  is the  $L_0$  constraint - this is the generalization of the mass shell condition for the interacting theory.
2. In the LV formalism we consider the same set of external physical fields conjugate to the same vertex operators. Assume that they obey the same conditions. There are also additional auxiliary fields and also gauge invariances. But these OC constraints are equivalent to the constraints on the LV fields after using gauge transformations to select a gauge. Then the  $\Sigma$  dependence drops out. Thus when the fields of the OC formalism obey physical state conditions, the LV vertex operator also has no  $\Sigma$  dependence and, in some gauge where the extra fields are gauge fixed to zero, is identical to the OC vertex operator. The  $L_0$  constraint is thus the same (i.e. gives the same equation) in both formalisms.
3. We can apply the above sequence of arguments to the interactive vertex

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<sup>1</sup>What enters in all the equations is the dimensionless ratio  $\frac{R}{a}$  where  $a$  is the world sheet cutoff. Setting  $R = 1$  is equivalent to rescaling  $a$ .

operators of the two formalisms: For the interacting case, in the OC formalism, the usual Ward identities ensure that if the external fields obey physical state conditions then integrated correlations involving gauge transformations ( $L_{-n}$ ) on any vertex operator (or set of vertex operators) is zero. This is equivalent to saying that the action of  $L_n$  on the rest of the vertex operators is zero (inside an integrated correlation function - the integration is important)<sup>2</sup>. This is thus the condition that the interacting vertex operators defined earlier in this section (by the substitution (5.0.62) is independent of  $\rho$  . But in that case so is the interacting LV vertex operator independent of  $\Sigma$  - since the algebraic manipulations relating the two are exactly the same in the free and interacting cases. Thus to summarize: If the external fields are physical, then the OC interacting vertex operator is  $\rho$  independent and the LV interacting vertex operator is  $\Sigma$  independent.

4. The equations of motion are obtained as the coefficients of a particular vertex operator. When the external fields obey the  $L_n$  constraints then what remains of the equation of motion is just the  $L_0$  condition for the interacting theory. This we have seen is the same in both formalisms. When we relax the  $L_n$  constraints on external fields and work in a general gauge, the equation of motion in the LV formalism picks up two kinds of terms:
  - i) terms proportional to the constraints themselves. This does not affect the physics, because we know that for physical external fields these terms are zero, and only physical fields are required for the S-matrix. So these terms cannot affect the S-matrix that is implied by these equations of motion.
  - ii) total derivatives in  $x_n$  - these are generated when we integrate by parts on  $x_n$ . These generate LV gauge transformations. This continues to be true for the interacting theory. For the free theory we saw that this is equivalent to the action of some linear combinations of  $L_{-n}$ 's on vertex operators. This must continue to be true for the interacting theory because the algebra is exactly the same. These terms also do not affect the S-matrix - this follows from the usual Ward identities of conformal field theory.

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<sup>2</sup>This is the statement that the state  $|\Psi\rangle = V_N \Delta V_{N-1} \Delta V_{N-2} \dots \Delta |\Phi_M\rangle$  where  $V_i$  are physical vertex operators,  $|\Phi_M\rangle$  is a physical state and  $\Delta$  are the propagators, satisfies  $L_n |\Psi\rangle = 0$ ,  $n > 0$ , provided also that  $|\Psi\rangle$  satisfies  $(L_0 - 1)|\Psi\rangle = 0$ .

5. Thus the S-matrix implied by the gauge invariant equations is equal to the S-matrix of the gauge fixed theory with physical external fields. This in turn is equal to that of the OC formalism.

## 6 Conclusions

This paper is a step in giving a precise mathematical basis for the connection between the loop variable formalism for string theory and some of the other formalisms. In this paper we have given the details of the correspondence between the old covariant formalism and the loop variable formalism for open bosonic string theory. At the free level the LV formalism has the same field content as BRST string field theory (in  $D=26$ ) and is also gauge invariant (in any dimension). The free theory is fully determined by the field content. Therefore one expects that there is a field redefinition in  $D=26$ , that would take one to the other, and also after gauge fixing, to the fields of the old covariant formalism. We verify this expectation for the old covariant formalism explicitly, by giving the map between the fields, constraints and gauge transformations for Level 2 and 3. (By gauge transformations we mean, in the old covariant formalism, the action of  $L_{-n}$ .) Such a map exists only in  $D=26$ . In other dimensions one can match constraints or gauge transformations, but not both.

In the loop variable approach, the interacting theory looks formally like a free theory with a different set of generalized loop variables. Therefore one can use the same techniques to show equivalence with the old covariant formulation of string theory. It is argued that the integrated correlation functions are the same in both formalisms as long as the physical state constraints are obeyed by the external fields. Thus the S-matrices are the same and they describe the same physical theory - at tree level. A rigorous proof can presumably be made along the lines above though we have not done so in this paper.

There remain many open questions. For instance one would like to understand the map to the BRST formalism. It is also not clear what the loop variable formalism describes away from  $D=26$ , in particular one can ask whether it describes a consistent non-critical string theory. Perhaps the most pressing question is whether one can use it in a quantitative way to get non trivial solutions to string theory. The man-

ifest background independence demonstrated (for gravitational backgrounds) in [4] may help in this regard.

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